Separación de variables en polares.

Ejemplo 1. Resolver el problema de Dirichlet

$$\begin{cases} \Delta u = 0 & \text{en el interior de disco radio } R_0 \\ u = f & \text{en la frontera} \end{cases}$$

$$\begin{cases} \Delta u = u''_{rr} + \frac{1}{r} u'_r + \frac{1}{r^2} u''_{\theta\theta} = 0 & 0 < r < R_0,\ -\pi < \theta < \pi \\ u(R_0, \theta) = f(\theta) & \\ u(r, -\pi) = u(r, \pi) & 0 < r < R_0 \\ u'_\theta(r, -\pi) = u'_\theta(r, \pi) & \end{cases}$$

Proponemos: $u(r,\theta) = R(r) \cdot T(\theta)$

En la ec. dif: $R''(r)T(\theta) + \frac{1}{r} R'(r) T(\theta) + \frac{1}{r^2} R(r) T''(\theta) = 0$

$$\frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)} = \lambda \qquad \lambda = cte$$

Para θ: $R(r)\,T(-\pi) = R(r)\,T(\pi) \Rightarrow T(-\pi) = T(\pi)$
$R(r)\,T'(-\pi) = R(r)\,T'(\pi) \Rightarrow T'(-\pi) = T'(\pi)$

Entonces:

$$\begin{cases} T''(\theta) + \lambda T(\theta) = 0 & -\pi < \theta < \pi \\ T(\pi) = T(-\pi) & \\ T'(-\pi) = T'(\pi) & \end{cases} \rightarrow \text{debe ser } \lambda \geq 0$$

Si $\lambda = 0$: $T(\theta) = t_0$

Si $\lambda > 0$, $\lambda = \alpha^2$: $T(\theta) = a\cos(\alpha\theta) + b\,\mathrm{sen}(\alpha\theta)$

Debe ser 2π-periódico : $\alpha = n \in \mathbb{N}$

Por otro lado:

$$r^2 R''(r) + r R'(r) - \lambda R(r) = 0 \qquad 0 < r < R_0$$

Proponemos soluciones $R(r) = r^p \qquad p \neq 0$

$$R'(r) = p\, r^{p-1}$$
$$R''(r) = p(p-1)\, r^{p-2}$$

Reemplazando:

$$p(p-1) r^p + p r^p - \lambda r^p = 0$$

$$p(p-1) + p - \lambda = 0$$

$$\boxed{p^2 = \lambda} \qquad \text{si } \lambda > 0$$

Como $\lambda = \alpha^2 = n^2 = p^2 \Rightarrow p = n \in \mathbb{N}$ si $\lambda > 0$

(no puede ser $p = -n$, porque $R(r) = r^{-n}$ no es continuo en $r = 0$)

si $\underline{\lambda = 0}$, la EDO: $r^2 R'' + r R' = 0$

$$\frac{R''}{R'} = -\frac{1}{r}$$

$$\ln |R'| = -\ln r + c$$

$$R' = r^{-1} . k$$

$$R(r) = k \ln r + h$$

Como queremos continuidad en el centro de círculo,

$\underline{k = 0}$

Entonces: si $\lambda = 0$: $u_0(r,\theta) = t_0 . h$

si $\lambda = n^2 > 0$: $u_n(r,\theta) = r^n \left(a_n \cos(n\theta) + b_n \operatorname{sen}(n\theta)\right)$

Superposición: $u(r,\theta) = t_0 . h + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \operatorname{sen}(n\theta)\right)$

Para $r = R_0$:

$$\boxed{u(R_0,\theta) = t_0 h + \sum_{n=1}^{\infty} R_0^n a_n \cos n\theta + R_0^n b_n \,\mathrm{sen}(n\theta) = f(\theta)}$$

$$\Rightarrow \boxed{\begin{aligned} t_0 h &= \frac{a_0}{2} = \frac{1}{2}\,\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta \\ R_0^n a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \\ R_0^n b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathrm{sen}(n\theta)\,d\theta \end{aligned}}$$

Ejemplo 2 (similar al 2se TP7)

$$u''_{rr} + \frac{1}{r}u'_r + \frac{1}{r^2}u''_{\theta\theta} = 0 \qquad 1 < r < 2, \quad -\pi < \theta < \pi$$

$u(1,\theta) = x$ $\quad -\pi < \theta < \pi$

$u(2,\theta) = y$

$u(r,-\pi) = u(r,\pi)$ $\quad 1 < r < 2$

$u'_\theta(r,-\pi) = u'_\theta(r,\pi)$

Haciendo separación de variables, llegamos:

$$\begin{cases} T''(\theta) + \lambda T(\theta) = 0 \\ T(\pi) = T(-\pi) \\ T'(\pi) = T'(-\pi) \end{cases} \qquad \begin{cases} r^2 R''(r) + r R'(r) - \lambda R(r) = 0 \\ 1 < r < 2 \end{cases}$$

$\downarrow$

$T_n(\theta) = a\cos(n\theta) + b\,\mathrm{sen}(n\theta)$

$T_0(\theta) = a$

$\downarrow$

$R_n(r) = c\,r^n + d\,r^{-n}$ si $\lambda \neq 0$

$R_0(r) = e \ln r + h$ si $\lambda = 0$

Entonces:

$$u_n(r,\theta) = a_n r^n \cos(n\theta) + b_n r^n \operatorname{sen}(n\theta) + c_n r^{-n}\cos(n\theta) + d_n r^{-n}\operatorname{sen}(n\theta)$$

$$u_0(r,\theta) = a_0 \ln r + b_0$$

Superposición:

$$u(r,\theta) = a_0 \ln r + b_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \operatorname{sen}(n\theta) + c_n r^{-n}\cos(n\theta) + d_n r^{-n}\operatorname{sen}(n\theta)$$

Condiciones de borde:

$r=1$:

$$u(1,\theta) = a_0 \ln 1 + b_0 + \sum_{n=1}^{\infty} (a_n + c_n)\cos(n\theta) + (b_n + d_n)\operatorname{sen}(n\theta) = x = 1.\cos\theta$$

$$\Rightarrow \begin{cases} b_n + d_n = 0 & \forall n \geq 1 \\ b_0 = 0 & \\ a_n + c_n = 0 & \text{si } n \geq 2 \\ a_1 + c_1 = 1 & \end{cases}$$

$r=2$.

$$u(2,\theta) = a_0 \ln 2 + b_0 + \sum \left(a_n 2^n + \frac{c_n}{2^n}\right)\cos(n\theta) + \left(b_n 2^n + \frac{d_n}{2^n}\right)\operatorname{sen}(n\theta) = y = 2\operatorname{sen}\theta$$

$$\Rightarrow \begin{cases} a_0 \ln 2 + b_0 = 0 & \\ 2^n a_n + \frac{c_n}{2^n} = 0 & n \geq 1 \\ 2^n b_n + \frac{d_n}{2^n} = 0 & n \geq 2 \\ 2b_1 + \frac{d_1}{2} = 2 & \end{cases}$$

$$\Rightarrow a_0 = b_0 = 0, \quad a_n = b_n = c_n = d_n = 0 \text{ si } \underline{n \geq 2}$$

Queda:

$$a_1 + c_1 = 1$$
$$2a_1 + \frac{c_1}{2} = 0 \quad \rightarrow \quad a_1 = -1/3, \quad c_1 = 4/3$$

$$b_1 + d_1 = 0$$
$$2b_1 + \frac{d_1}{2} = 2 \quad \rightarrow \quad b_1 = \frac{4}{3}, \quad d_1 = -\frac{4}{3}$$

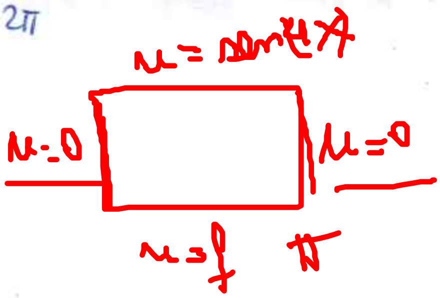
Solución:

$$u(r,\theta) = \left(-\frac{1}{3}r + \frac{4}{3r}\right)\cos\theta + \left(\frac{4}{3}r - \frac{4}{3r}\right)\operatorname{sen}\theta$$

Ejemplo 3 (de integrador 5/2/21)

Resolver:

$$\Delta u(x,y) = 0 \qquad 0<x<\pi,\ 0<y<2\pi$$
$$u(0,y) = u(\pi,y) = 0 \qquad 0<y<2\pi$$
$$u(x,0) = f(x) \qquad 0<x<\pi$$
$$u(x,2\pi) = \operatorname{sen}(4x) \qquad 0<x<\pi$$



Dividimos el problema:

I

$$\begin{cases} \Delta u(x,y) = 0 \\ u(0,y) = u(\pi,y) = 0 \\ u(x,0) = 0 \\ u(x,2\pi) = \operatorname{sen}(4x) \end{cases}$$

II

$$\begin{cases} \Delta u(x,y) = 0 \\ u(0,y) = u(\pi,y) = 0 \\ u(x,0) = f(x) \\ u(x,2\pi) = 0 \end{cases}$$

Proponemos $u(x,y) = X(x)\,Y(y)$

ED: $$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

Resulta: $\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X(\pi) = 0 \end{cases}$

Debe ser $\lambda = n^2$, $X_n(x) = a_n \operatorname{sen}(nx)$

En I: $\begin{cases} Y''(y) - n^2 Y(y) = 0 \\ Y(0) = 0 \end{cases}$

$$Y_n(y) = c_n e^{ny} + d_n e^{-ny} = \tilde{c}_n \operatorname{sh}(ny) + \tilde{d}_n \operatorname{ch}(ny)$$

Con $y = 0$: $Y_n(0) = \tilde{d}_n = 0$

$$\Rightarrow Y_n(y) = \tilde{c}_n \operatorname{sh}(ny)$$

Luego: $u(x,y) = \sum_1^\infty X_n(x) Y_n(y) = \sum_1^\infty A_n \operatorname{sen}(nx) \operatorname{sh}(ny)$

con $y = 2\pi$:

$$u(x, 2\pi) = \sum A_n \operatorname{sen}(nx) \operatorname{sh}(2n\pi) = \operatorname{sen}(4x)$$

$$\Rightarrow A_n = 0 \quad n \geq 1, \ n \neq 4$$

$$A_4 \operatorname{sh}(8\pi) = 1 \longrightarrow A_4 = \frac{1}{\operatorname{sh}(8\pi)}$$

$$\boxed{u(x,y) = \frac{1}{\operatorname{sh}(8\pi)} \operatorname{sen}(4x) \operatorname{sh}(4y)}$$

En II: $\begin{cases} Y''(y) - n^2 Y(y) = 0 \\ Y(2\pi) = 0 \end{cases}$

$$Y_n(y) = c_n e^{ny} + d_n e^{-ny} = \tilde{c}_n \operatorname{sh}(ny) + \tilde{d}_n \operatorname{ch}(ny)$$

con $y = 2\pi$: $Y_n(2\pi) = \tilde{c}_n \operatorname{sh}(2\pi n) + \tilde{d}_n \operatorname{ch}(2\pi n) = 0$

$$\Rightarrow \quad \tilde{d}_n = - \tilde{c}_n \frac{sh(2\pi n)}{ch(2\pi n)}$$

$$Y_n(y) = \tilde{c}_n \left(sh(ny) - \frac{sh(2\pi n)}{ch(2\pi n)} ch(ny) \right) =$$

$$= \tilde{c}_n \left(sh(ny)\, ch(2\pi n) - sh(2\pi n)\, ch(ny) \right) / ch(2\pi n)$$

$$= \tilde{c}_n \; sh(ny - 2\pi n) / ch(2\pi n)$$

Luego: $$u(x,y) = \sum_{1}^{\infty} X_n(x)\, Y_n(y)$$

$$\boxed{= \sum_{1}^{\infty} A_n \, sen(nx) \cdot \frac{sh(ny - 2\pi n)}{ch(2\pi n)}}$$

con $y=0$: $$u(x,0) = \sum_{1}^{\infty} \underbrace{A_n \frac{sh(-2\pi n)}{ch(2\pi n)}}_{\text{coef fourier de f de serie de senos}} sen(nx) = f(x)$$

$\Rightarrow$

$$A_n \frac{sh(-2\pi n)}{ch(2\pi n)} = \frac{1}{\pi} \int_0^{\pi} f(x)\, sen(nx)\, dx$$

Solución al prob. dado: $$\boxed{u(x,y) = \frac{sen(4x)\, sh(4y)}{sh(8\pi)} + \sum_{1}^{\infty} A_n \, sen\, nx \, \frac{sen(ny - 2\pi n)}{ch(2\pi n)}}$$